Rigid Body Hybrid Dynamics

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In the Hybrid Dynamical Systems chapter we set up a general framework for defining these systems. Here, we fill in the details for each component of the hybrid dynamical system for rigid body dynamics with plastic impact. We can then use this system for general manipulation or self-manipulation problems. This chapter is based largely on [1, Def. 5].

First, a reminder about how we have defined our set of contact constraints and modes. The set of possible contact constraints is $\mathcal{K} = \{i, j, \dots, k\}$, which we can partition into the normal (non-penetration constraints) and tangential (frictional constraints) directions as $\mathcal{K} = \mathcal{K}_n \cup \mathcal{K}_t$. A given contact mode is $\mathrm{I}, \mathrm{J}, \mathrm{K}, \ldots \subseteq \mathcal{K}$, which we may also want to partition into normal and tangential constraints, $\mathrm{I}_n := \mathrm{I} \cap \mathcal{K}_n$ and $\mathrm{I}_t := \mathrm{I} \cap \mathcal{K}_t$. Finally, we may need to determine the *corresponding normal* constraint to a given tangential constraint (i.e. the normal direction constraint of the same contact pair), which we call $\alpha(i_t) = i_n$. A position constraint (in the normal direction) is $a_i(q) = 0$, while a velocity constraint is $A_i \dot{q} = 0$.

1 Rigid Body Hybrid Dynamics Definition

The hybrid dynamical system for a collection of rigid bodies subject to impacts is a hybrid dynamical system $\mathcal{H} := (\mathcal{J}, \Gamma, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$ with each component defined as follows.

1.1 Modes

The set of discrete modes is determined by the different contact conditions,

$$\mathcal{J} := \{ \underbrace{\mathrm{I} \subseteq \mathcal{K}}_{\text{Contact mode}} : \underbrace{\exists\, q \in \mathcal{Q} \text{ s.t. } a_{\mathrm{I}_n}(q) = 0, a_{\mathcal{K}_n}(q) \geq 0,}_{\substack{\text{A point exists that} \\ \text{satisfies position constraints}}} \underbrace{\alpha(\mathrm{I}) \subseteq \mathrm{I}}_{\substack{\text{Corresponding} \\ \text{normals}}} \} \tag{1}$$

There are three components to this definition. First, each mode I is a subset of the set of possible contact constraints $\mathcal{K}$. Then, there must be at least one point in the configuration space that satisfies the position constraint a_{I_n} that are active in I as well as all of the other inequality constraints $a_{\mathcal{K}_n}$ (i.e. the domain D_{I}, defined below, is not empty). Finally, we require for each tangential constraint that the corresponding normal constraint is also included. Having a frictional constraint without touching the surface wouldn't make much sense.

1.2 Transitions

Out of all possible transitions, we limit our transition set to be,

$$\Gamma := \{ \underbrace{(\mathrm{I}, \mathrm{J}) \in \mathcal{J} \times \mathcal{J}}_{\text{All possible transitions}} : \underbrace{\mathrm{I} \neq \mathrm{J},}_{\substack{\text{No self-transitions} \\ \text{(plastic impact)}}} \underbrace{\mathrm{I} \cup \mathrm{J} \in \mathcal{J}}_{\substack{\text{The combined} \\ \text{mode is valid}}} \} \tag{2}$$

Here, each transition is written as an ordered pair (I,J) of modes, each of which are pulled from $\mathcal{J}$, representing a transition from I to J. In the case of plastic impact, we disallow self-transitions. For elastic impact, it is sometimes helpful to allow self transitions, but we focus on plastic impact in this chapter.

The final condition is that for transition (I,J) to be valid, the combined set of constraints from both I and J must also be a valid mode in $\mathcal{J}$. In many cases we are just adding or removing constraints and so either $\mathrm{I} \subset \mathrm{J}$ or $\mathrm{J} \subset \mathrm{I}$. In those cases this is condition is trivially satisfied. The more interesting case is if we are both adding and removing constraints in one transition, which typically happens when impacting a new constraint causes an existing constraint to lift off. In that case, the mode where all of the constraints are kept simultaneously must be valid even if it isn't what is happening here. Looking at the conditions on $\mathcal{J}$ in (1), only the position constraint condition must be checked for this combined mode (as $\mathrm{I} \cup \mathrm{J} \subseteq \mathcal{K}$ and $\alpha(\mathrm{I} \cup \mathrm{J}) \subseteq \mathrm{I} \cup \mathrm{J}$ are true by construction). Thus, for a transition to be valid, there must be some point q that satisfies the position constraints of all contacts in question, i.e. there is a point where it is possible to "touch" all constraints simultaneously, and thus a point where this transition could possibly occur.

Remember that Γ does not consider the dynamics of if a transition is achievable, rather it just gives the adjacency structure for the modes. For this reason, even though Γ contains ordered pairs, both directions will always be included, i.e. if $(\mathrm{I},\mathrm{J}) \in \Gamma$ then $(\mathrm{J},\mathrm{I}) \in \Gamma$. For achievable transitions, which depend on the dynamics, we must consider $\tilde{\Gamma} := \{(\mathrm{I},\mathrm{J}) \in \Gamma : G_{\mathrm{I},\mathrm{J}} \neq \varnothing\}$.

1.3 Domains

Each domain D_I is defined in terms of an overall state space $(q,\dot{q}) \in T\mathcal{Q}$ and enforces the constraints of that mode,

$$D_\mathrm{I} := \{ \underbrace{(q,\dot{q}) \in T\mathcal{Q}}_{\text{Overall state space}} : \underbrace{a_{\mathrm{I}_n}(q) = 0,}_{\substack{\text{Active position}\\ \text{constraints}}} \quad \underbrace{A_\mathrm{I}\dot{q} = 0,}_{\substack{\text{Active velocity}\\ \text{constraints}}} \quad \underbrace{a_{\mathcal{K}_n}(q) \geq 0}_{\substack{\text{All position}\\ \text{constraints}}} \} \tag{3}$$

For a point to be part of the domain for mode I, it must satisfy the position and velocity constraints of that mode. Note that we only have position constraints in the normal direction (frictional/tangential constraints are on velocity). Furthermore, for a point to be a valid part of a given mode, it must also satisfy the inequality position constraint of all constraints, including the inactive ones. These encode, e.g., that a point must not lie below the ground.

1.4 Flows

The flow or vector field must enforce the dynamics for that mode, which we have found to be,

$$F_\mathrm{I}\left(\begin{bmatrix} q \\ \dot{q} \end{bmatrix}\right) := \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M_\mathrm{I}^\dagger(\Upsilon - C\dot{q} - N) - A_\mathrm{I}^{\dagger T}\dot{A}_\mathrm{I}\dot{q} \end{bmatrix} \tag{4}$$

Here we are using the block matrix formulation of the dynamics, where recall that the definition of $M_\mathrm{I}^\dagger$, $A_\mathrm{I}^{\dagger T}$, and A_I depend on the contact mode I. As noted in the hybrid system chapter, the flow is also a function of control input $\Upsilon(u)$, and may also be time-varying. If we need to make these dependencies explicit we can write $F_\mathrm{I}(t,q,\dot{q},u)$.

1.5 Guards

The guard set is the trickiest part of defining this hybrid system. At a high level, there are two cases to consider: if we are impacting a new constraint, in which case we must follow impulse-velocity complementarity, or if we are not, in which case we must follow force-acceleration complementarity. Specifically,

$$G_{\mathrm{I},\mathrm{J}} = \Bigg\{ \overbrace{(q,\dot{q}) \in D_\mathrm{I}}^{\text{Points in domain I}} : \text{If } \overbrace{\exists\, k \in \mathcal{K}_n : a_k(q) = 0,\ A_k\dot{q} < 0}^{\text{If there is a new impact}} \quad \begin{array}{ll} \text{Then} & \overbrace{U_\mathrm{J}(\hat{P}_\mathrm{J}) \geq 0,\ A_i\dot{q}_\mathrm{J}^+ > 0}^{\text{Impulse-Velocity Conditions}} \quad \overbrace{\forall\, i \in \mathcal{I}_{IV} \setminus \mathrm{J},}^{\text{IV Scope}} \\ \text{Otherwise} & \underbrace{U_\mathrm{J}(\lambda_J) \succeq 0, A_i\ddot{q} + \dot{A}_i\dot{q} \succ 0}_{\text{Force-Acceleration Conditions}} \quad \underbrace{\forall\, i \in \mathcal{I}_{FA} \setminus \mathrm{J}}_{\text{FA Scope}} \end{array} \Bigg\} \tag{5}$$

where recall that the *scope* defines which contact constraints must be considered in the complementarity conditions,

$$\mathcal{I}_{IV}(q,\dot{q}) = \{i \in \mathcal{K} : a_{\alpha(i)}(q) = 0\} \tag{6}$$

$$\mathcal{I}_{FA}(q,\dot{q}) = \{i \in \mathcal{K} : a_{\alpha(i)}(q) = 0, A_i(q)\dot{q} = 0, A_{\alpha(i)}(q)\dot{q} = 0\} \tag{7}$$

The scope consists of all constraints which the current state is touching, while for force-acceleration we also require velocity constraint satisfaction. This allows, among other things, frictional slip-to-stick transitions. Numerically, the scope can be limited to just the current constraints I and any potential new constraints from impact or friction[1]. See the Complementarity chapter for further details.

The dependence of the guard set on the dynamics for the trending conditions in the force-acceleration complementarity means that the guard sets, like the flows, are a function of control input u and may be time-varying, thus we can write $G_{\mathrm{I,J}}(t,u)$.

In this guard set definition, we only need to check the inequality constraints from complementarity, as the equality constraints are enforced by the reset map (for impulse-velocity) or the flow (for force-acceleration). Note that we could have equally used any of the equivalent formulations of the complementarity conditions as listed in the Complementarity chapter.

While defining the exact guard set is somewhat involved, defining the outlet set (points which will cause a reset to any different mode) is much easier,

$$G_{\mathrm{I}} = \{(q,\dot{q}) \in D_{\mathrm{I}} : \underbrace{\exists\, k \in \mathcal{K}_n : a_k(q) \prec 0}_{\text{Impact}}\} \quad \cup \quad \{(q,\dot{q}) \in D_{\mathrm{I}} : \underbrace{\exists\, i \in I : U_i(\lambda_{\mathrm{I}}) \prec 0}_{\text{Liftoff}}\} \tag{8}$$

That is, the outlet set consists of all points where we have to add a constraint and all points where we must remove a constraint. Comparing (5) to (8), note that the trending condition $a_k(q) \prec 0$ can be broken down into $a_k(q) < 0$, which is disallowed in the definition of D_{I} from (3), or $a_k(q) = 0$ and $A_k\dot{q} < 0$, as in the impact condition of (5), or that acceleration or higher order is trending negative, which would trigger the force-acceleration condition of (5).

1.6 Resets

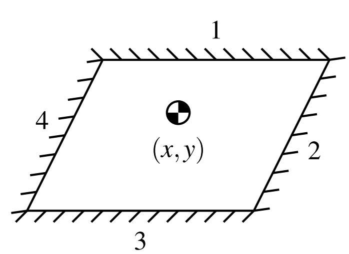
The reset map encodes our impact law, which here we take to be plastic impact,

$$R_{\mathrm{I,J}}\left(\begin{bmatrix} q \\ \dot{q}^- \end{bmatrix}\right) := \begin{bmatrix} q \\ \dot{q}^- - A_{\mathrm{J}}^{\dagger T} A_{\mathrm{J}} \dot{q}^- \end{bmatrix} = \begin{bmatrix} q \\ \dot{q}^+ \end{bmatrix} \tag{9}$$

Note that this reset map depends only on the next mode J and not the prior mode I. Also note that for liftoff transitions, we do not need to apply a reset map as all constraints should already be satisfied. However, we can apply this same reset map everywhere since this should have no effect for liftoff transitions, and will actually correct for any numerical drift in our velocity constraints.

2 Example

Consider a simple example of a point particle of mass m constrained by four plastic, frictionless constraints, $\mathcal{K} = \{1,2,3,4\}$, as shown at right. The constraint equations are,



$$a_1(x,y) = -y + 1 \geq 0 \qquad a_2(x,y) = -2x + y + 3 \geq 0 \tag{10}$$

$$a_3(x,y) = y + 1 \geq 0 \qquad a_4(x,y) = 2x - y + 3 \geq 0 \tag{11}$$

Assume that we can apply control in both directions, $u = [u_x, u_y]^T$, and for now we can ignore gravity or other external forces.

[1] Note that the scope of force-acceleration problem $\mathcal{I}_{FA}$ considers constraints which are touching but with no relative velocity. Thus, if the constraint were tangent to the motion but not already active in I, it could be added by the force-acceleration complementarity. However, this is not possible in numerical simulation as it requires exactly matching the position and velocity simultaneously.

Modes: The set of possible modes is not every subset of $\mathcal{K}$, since the point cannot touch 1 and 3 or 2 and 4 simultaneously. Thus,

$$\mathcal{J} = \{0,1,2,3,4,12,14,23,34\} \tag{12}$$

where we will define $0 := \{\}$ as the mode with no constraints active, $12 := \{1,2\}$ as the mode with both constraints 1 and 2 active, etc, and always listed in numerical order. In particular, note that $13 := \{1,3\} \subset \mathcal{K}$ but $13 \notin \mathcal{J}$, and similarly for 24, 123, 234, 134, 124, and 1234.

Transitions: From this, the set of edges is,

$$\begin{array}{lll}\Gamma = \{ & (0,1),(0,2),(0,3),(0,4), & (0,12),(0,14),(0,23),(0,34), \\ & (1,0),(1,2),(1,4),(1,12),(1,14), & (12,0),(12,1),(12,2), \\ & (2,0),(2,1),(2,3),(2,12),(2,23), & (14,0),(14,1),(14,4), \\ & (3,0),(3,2),(3,4),(3,23),(3,34), & (23,0),(23,2),(23,3), \\ & (4,0),(4,1),(4,3),(4,14),(4,34), & (34,0),(34,3),(34,4) \quad \} \end{array} \tag{13}$$

$$\begin{tikzcd} 14 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & 1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & 12 \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ 4 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & 0 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & 2 \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ 34 \arrow[r, leftrightarrow] & 3 \arrow[r, leftrightarrow] & 23 \end{tikzcd}$$

There are many interesting properties of this set:

First, note the combinatorial growth in the number of possible transitions: we have $|\mathcal{K}| = 4$, $|\mathcal{J}| = 9$, and $|\Gamma| = 40$. In general, $|\mathcal{J}| \leq 2^{|\mathcal{K}|}$ and $|\Gamma| \leq |\mathcal{J}|^2 - |\mathcal{J}|$ (since no self-transitions are allowed).

Second, note that both directions of each transition are included at this point, as we have not considered the dynamics. If we were to say add gravity, then possibly $(0,3)$ would be achievable but maybe not $(3,0)$.

Third, note that this set includes some "double transitions" such as $(0,12)$, which requires the point to hit both constraints simultaneously, or its reverse $(12,0)$, which requires both constraints to release simultaneously. These are virtually impossible to achieve numerically, but included here for completeness.

Finally, it is also interesting to consider which transitions are not here. From the definition in (2), in particular the condition that $\mathrm{I} \cup \mathrm{J} \in \mathcal{J}$, we see that transitions like $(1,3)$ are excluded since $13 \notin \mathcal{J}$, and similarly for $(1,23)$, $(12,34)$, etc.

Domains: The domains of this system, after simplifying form the generic definition in (3), are,

$$D_0 = \{(x,y,\dot{x},\dot{y}) : -3 \leq 2x - y \leq 3,\ -1 \leq y \leq 1\} \tag{14}$$

$$D_1 = \{(x,y,\dot{x},\dot{y}) : -1 \leq x \leq 2,\ y = 1,\ \dot{y} = 0\} \qquad D_2 = \{(x,y,\dot{x},\dot{y}) : 1 \leq x \leq 2,\ y = 2x - 3,\ \dot{y} = 2\dot{x}\} \tag{15}$$

$$D_3 = \{(x,y,\dot{x},\dot{y}) : -2 \leq x \leq 1,\ y = -1,\ \dot{y} = 0\} \qquad D_4 = \{(x,y,\dot{x},\dot{y}) : -2 \leq x \leq -1,\ y = 2x + 3,\ \dot{y} = 2\dot{x}\} \tag{16}$$

$$D_{12} = \{(x,y,\dot{x},\dot{y}) = (2,1,0,0)\} \qquad D_{14} = \{(x,y,\dot{x},\dot{y}) = (-1,1,0,0)\} \tag{17}$$

$$D_{23} = \{(x,y,\dot{x},\dot{y}) = (1,-1,0,0)\} \qquad D_{34} = \{(x,y,\dot{x},\dot{y}) = (-2,-1,0,0)\} \tag{18}$$

where note the doubly-constrained domains reduce to a single point.

Recall from (3) that even if no contacts are active, we still need all four inequality constraints to ensure we are "inside" the domain. As constraints become active, they switch to being equality constraints while the others remain, and we add an equality constraint on velocity.

Flows: For the dynamics, assume that there is no gravity or external forces other than the control input. Then plugging into (4) we have,

$$F_0 = \frac{d}{dt}\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ u_x/m \\ u_y/m \end{bmatrix}, \quad F_1 = F_3 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ u_x/m \\ 0 \end{bmatrix}, \quad F_2 = F_4 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ u_x/5m + 2u_y/5m \\ 2u_x/5m + 4u_y/5m \end{bmatrix}, \quad \begin{array}{l} F_{12} = F_{14} = \\ = F_{23} = F_{34} = \end{array} \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

Note that in this case, because the constraints are parallel, the dynamics in modes 1 and 3 are equal and the same goes for 2 and 4 – this will not be true in general. We could also simplify these expressions further based on the domain, e.g. we know that $\dot{y} = 0$ in several domains.

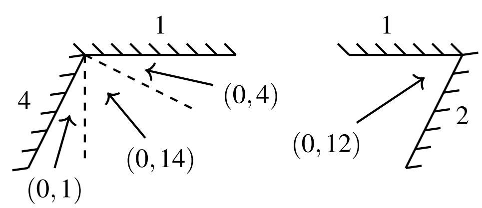
Guards: Each of the 40 edges in the transition set Γ has a unique guard set, and that guard depends on the dynamics and therefore the control input (and, if any, external forces like gravity). Each of these sets may have multiple components based on if there is an impulsive transition (in which case we follow impulse-velocity complementarity) or not (in which case we follow force-acceleration complementarity) and what other constraints the point is touching. Here, we will analyze all 10 guards involving mode 1, with the remaining guards defined analogously.

Guard (0,1) may seem at first like a simple transition, adding a single constraint, and indeed we will show that for numerical simulation we only need that $y = 1$ and $\dot{y} > 0$. However, the full definition must consider six cases:

$$\begin{aligned}
G_{0,1} = \{(x,y,\dot{x},\dot{y}) \in D_0 : \quad & a_1(x,y) = 0, && && (20)\\
\text{Impulsive cases:} \quad & \text{If } A_1\dot{q} < 0, a_2(x,y) > 0, a_4(x,y) > 0, && \text{Then } U_1(\hat{P}_1) \geq 0 && (21)\\
& \text{If } A_1\dot{q} < 0, a_2(x,y) = 0, && \text{Then } U_1(\hat{P}_1) \geq 0, A_2\dot{q}_1^+ > 0 && (22)\\
& \text{If } A_1\dot{q} < 0, a_4(x,y) = 0, && \text{Then } U_1(\hat{P}_1) \geq 0, A_4\dot{q}_1^+ > 0 && (23)\\
\text{Non-impulsive cases:} \quad & \text{If } A_1\dot{q} = 0, a_2(x,y) > 0, a_4(x,y) > 0, && \text{Then } U_1(\lambda_1) \succeq 0 && (24)\\
& \text{If } A_1\dot{q} = 0, a_2(x,y) = 0, A_2\dot{q} = 0, && \text{Then } U_1(\lambda_1) \succeq 0, A_2\ddot{q} + \dot{A}_2\dot{q} \succ 0 && (25)\\
& \text{If } A_1\dot{q} = 0, a_4(x,y) = 0, A_4\dot{q} = 0, && \text{Then } U_1(\lambda_1) \succeq 0, A_4\ddot{q} + \dot{A}_4\dot{q} \succ 0 \quad \} && (26)
\end{aligned}$$

Lets break down this guard set line-by-line. First note that overall, the guard is a subset of the starting domain and that we must be touching the constraints of the ending domain (20). Then, we have the most common case: if the system impacts constraint 1 with a non-zero velocity and without simultaneously impacting any other constraints, (21). Note that this is the only case where we don't have additional equality constraints – in numerical simulation, it is virtually impossible to reach zero on multiple constraints simultaneously. We can further simplify this condition by noting that the impulse will always satisfy the impulse-velocity condition in this case (as is always true for simple impacting constraints), since we know that for this case $A_1\dot{q} = -\dot{y} < 0$ and therefore $U_1(\hat{P}_1) = m\dot{y} > 0$. Thus, for numerical computation, this guard set (20)–(26) simplifies down to just,

$$G_{0,1} \approx \{(x,y,\dot{x},\dot{y}) \in D_0 : a_1(x,y) = 0, A_1\dot{q} < 0\} = \{(x,y,\dot{x},\dot{y}) \in D_0 : y = 1, \dot{y} > 0\} \tag{27}$$



The remaining cases, (22)–(26), are included for completeness and analysis, but are not necessary for simulation. They may be achievable, to within solver tolerance, in a trajectory optimization context.

Cases (22) and (23) consider the case where the system additionally impacts constraints 2 or 4 at the same moment of impact with constraint 1 (see at right). First consider the case of colliding with constraints 1 and 4 simultaneously. In order to end up in constraint 1 as the next mode, the approach velocity would have to be such that after impact with constraint 1 the resulting velocity is away from constraint 4 ($A_4\dot{q}_1^+ > 0$). We can see from the geometry of that corner that this will be true if $\dot{x}^- > 0$ at the time of impact. Meanwhile, there is no approach velocity into a collision with both constraints 1 and 2 that will result in the system ending in mode 1, and so (22) is not possible.

The non-impulsive cases, (24)–(26), are only possible if we have arrived at a position of $a_1 = 0$ with a velocity $A_1\dot{q} = 0$, for instance if the system was slowing down as it approached such that it reach zero velocity at that moment (but that we did not continue to accelerate downwards and away from the constraint, otherwise we would not satisfy the force condition $U_1(\lambda_1) \succeq 0$). Among these, we have the same three configurations as the impulsive case depending on whether we are additionally, simultaneously touching constraints 2 or 4 (again with no relative velocity). Unlike the impulsive case, it is theoretically possible to make the transition from 0 to 1 at the corner touching constraint 2 if the forces are such that the system is accelerating away from constraint 2 at that moment. However, these non-impulsive simultaneous transitions require four simultaneous equality conditions (that the position and velocity for both constraints reach zero all at the same time), such that they are not important for simulation.

Guard (1,12) is a good bit simpler than (0,1) even though both add just a single constraint,

$$
\begin{aligned}
G_{1,12} = \{(x,y,\dot{x},\dot{y}) \in D_1 : \quad & a_2(x,y) = 0, && && (28)\\
\text{Impulsive case:} \quad & \text{If } A_2\dot{q} < 0, && \text{Then } U_1(\hat{P}_{12}) \geq 0, U_2(\hat{P}_{12}) \geq 0 && (29)\\
\text{Non-impulsive case:} \quad & \text{If } A_2\dot{q} = 0, && \text{Then } U_1(\lambda_{12}) \succeq 0, U_2(\lambda_{12}) \succeq 0 \quad \} && (30)
\end{aligned}
$$

Here, the impulse condition will be satisfied always, while the force conditions depend on the applied forces (and require arriving at constraint 2 with exactly zero velocity), and so for simulation we can approximate this guard as,

$$G_{1,12} \approx \{(x,y,\dot{x},\dot{y}) \in D_1 : a_2(x,y) = 0, A_2\dot{q} < 0\} = \{(x,y,\dot{x},\dot{y}) : x = 2, y = 1, \dot{x} > 0, \dot{y} = 0\} \tag{31}$$

Guard (1,14) would appear to be quite similar to (1,12), however the geometry of the corner makes this transition much harder. The general form is the same,

$$
\begin{aligned}
G_{1,14} = \{(x,y,\dot{x},\dot{y}) \in D_1 : \quad & a_4(x,y) = 0, && && (32)\\
\text{Impulsive case:} \quad & \text{If } A_4\dot{q} < 0, && \text{Then } U_1(\hat{P}_{14}) \geq 0, U_4(\hat{P}_{14}) \geq 0 && (33)\\
\text{Non-impulsive case:} \quad & \text{If } A_4\dot{q} = 0, && \text{Then } U_1(\lambda_{14}) \succeq 0, U_4(\lambda_{14}) \succeq 0 \quad \} && (34)
\end{aligned}
$$

however there is no way for the impulse condition $U_1(\hat{P}_{14}) \geq 0$ to be satisfied. The only way for the system to achieve the (1,14) transition is to arrive with zero velocity,

$$G_{1,14} = \{(x,y,\dot{x},\dot{y}) \in D_1 : a_4(x,y) = 0, A_4\dot{q} = 0, U_1(\lambda_{14}) \succeq 0, U_4(\lambda_{14}) \succeq 0\} \approx \varnothing \tag{35}$$

which for numerical simulation can ignored.

Guard (1,4) is instead the likely transition from 1 if the state reaches constraint 4, and involves both adding and removing a constraint in one step,

$$
\begin{aligned}
G_{1,4} = \{(x,y,\dot{x},\dot{y}) \in D_1 : \quad & a_4(x,y) = 0, && && (36)\\
\text{Impulsive case:} \quad & \text{If } A_4\dot{q} < 0, && \text{Then } U_4(\hat{P}_{14}) \geq 0, A_1\dot{q}_4^+ > 0 && (37)\\
\text{Non-impulsive case:} \quad & \text{If } A_4\dot{q} = 0, && \text{Then } U_4(\lambda_4) \succeq 0, A_1\ddot{q} + \dot{A}_1\dot{q} \succ 0 \quad \} && (38)
\end{aligned}
$$

The impulse and separating velocity conditions will always be satisfied by construction if we approach with non-zero velocity, thus for simulation we can approximate the guard as (similar to $G_{1,12}$),

$$G_{1,4} \approx \{(x,y,\dot{x},\dot{y}) \in D_1 : a_4(x,y) = 0, A_4\dot{q} < 0\} = \{(x,y,\dot{x},\dot{y}) : x = -1, y = 1, \dot{x} < 0, \dot{y} = 0\} \tag{39}$$

For the non-impulsive case, the decision between (1,4) and (1,14) (or remaining in mode 1) depends on the forces.

Guard (4,1) is similar to (1,4) with the constraints reversed.

Guard (1,2) is similar to (1,14), where we have that $G_{1,2} \approx \varnothing$, as it is only possible non-impulsively if we arrive with zero velocity and the forces push us onto 2.

Guard (2,1) is similar to (1,2) and (1,14), where we have that $G_{2,1} \approx \varnothing$, as it is only possible non-impulsively if we arrive with zero velocity and the forces push us onto 1.

Guard (1,0) is a pure liftoff guard, where there are no new impacting constraints, so we only need to keep track of the current constraint and ensure no new ones are added,

$$
\begin{aligned}
G_{1,0} = \{(x,y,\dot{x},\dot{y}) \in D_1 : \quad & A_1\ddot{q} + \dot{A}_1\dot{q} \succ 0, && && (40)\\
& \text{If } a_2(x,y) = 0, A_2\dot{q} = 0, && \text{Then } A_2\ddot{q} + \dot{A}_2\dot{q} \succ 0 && (41)\\
& \text{If } a_4(x,y) = 0, A_4\dot{q} = 0, && \text{Then } A_4\ddot{q} + \dot{A}_4\dot{q} \succ 0 \quad \} && (42)
\end{aligned}
$$

Note that it may be more convenient to replace the acceleration conditions with a force condition, e.g. $U_1(\lambda_1) \prec 0$ – if we were to keep the constraint, it would violate our force conditions (see the Complementarity chapter).

In (1,0), the only difficulty is what happens in the corners of domain D_1 – if we are touching (but not impacting, i.e. with no relative velocity) the corners, then it depends on the dynamics whether we will transition into mode 0 or another mode (namely, 2 or 4). In particular, compare the conditions in $G_{1,0}$ to that in $G_{1,4}$, especially (38).

Guard (12,1) is also a liftoff guard, however one where we are not removing all active constraints,

$$G_{12,1} = \{(x,y,\dot{x},\dot{y}) \in D_{12} : U_1(\lambda_1) \succeq 0, A_2\ddot{q} + \dot{A}_2\dot{q} \succ 0\} = \{(x,y,\dot{x},\dot{y}) \in D_{12} : U_1(\lambda_1) \succeq 0, U_2(\lambda_{12}) \prec 0\} \tag{43}$$

Note that here we need to check that constraint 1 is still valid while constraint 2 is no longer valid.

Guard (14,1) is similar to (12,1) but with constraint 4 instead of 2.

Outlet Set G_1: The outlet set (points which will cause a reset to any different mode) is generally easier to define than the individual guards, as we don't have to differentiate, e.g., between the (1,2) and (1,12) transitions. For mode 1,

$$G_1 = G_{1,0} \cup G_{1,2} \cup G_{1,4} \cup G_{1,12} \cup G_{1,14} = \{(x,y,\dot{x},\dot{y}) \in D_1 : a_2(x,y) \prec 0 \ \vee \ a_4(x,y) \prec 0 \ \vee \ U_1(\lambda_1) \prec 0\} \tag{44}$$

thus if constraints 2 or 4 are impacting, or constraint 1 is releasing, we must transition somewhere. (Recall that $\vee$ is a logical "or").

Resets: Defining the reset map is considerably easier than the guard sets. All reset maps can follow the general form (9) of applying the impact law onto the resulting domain (as it will have no effect for non-impulsive conditions).

But we can simplify the definitions here by plugging in for the matrices $A_J^{\dagger T} A_J$ and applying the constraints of the starting mode. For brevity we will again, without loss of generality, limit the explicit definitions to those involving mode 1. Starting with the liftoff and non-impulsive transitions, which are all identity,

$$R_{1,0}\left(\begin{bmatrix} x \\ y \\ \dot{x}^- \\ \dot{y}^- \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ \dot{x}^- \\ \dot{y}^- \end{bmatrix} = \begin{bmatrix} x \\ 1 \\ \dot{x}^- \\ 0 \end{bmatrix} \qquad R_{12,1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad R_{14,1} = R_{1,14} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad R_{1,2} = R_{2,1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{45}$$

The remaining reset maps follow the impact law for the resulting mode. For example, when impacting into mode 1, $\dot{q}^+ = \dot{q}^- - A_1^{\dagger T} A_1 \dot{q}^- = [\dot{x}^-, 0]^T$. The full set of reset maps for mode 1 are,

$$R_{0,1}\left(\begin{bmatrix} x \\ y \\ \dot{x}^- \\ \dot{y}^- \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ \dot{x}_1^+ \\ \dot{y}_1^+ \end{bmatrix} = \begin{bmatrix} x \\ 1 \\ \dot{x}^- \\ 0 \end{bmatrix} \qquad R_{4,1} = \begin{bmatrix} -1 \\ 1 \\ \dot{x}^- \\ 0 \end{bmatrix} \qquad R_{1,4} = \begin{bmatrix} -1 \\ 1 \\ \dot{x}^-/5 \\ 2\dot{x}^-/5 \end{bmatrix} \qquad R_{1,12} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

3 Practice Problems

1) Consider the example from Section 2. What are the guard sets and reset functions for transitions (2,3), (3,2) (4,3), and (0,23)?

2) Consider the example from Section 2. If the only forces acting on the particle were gravity and contact (so, $u_x = 0$ and $u_y = -mg$), which of the transitions in (13) would be achievable? (I.e., have a non-empty guard set). You may assume the state is initialized in mode 0 but with any position or velocity.

References

[1] A. M. Johnson, S. E. Burden, and D. E. Koditschek, “A hybrid systems model for simple manipulation and self-manipulation systems,” *International Journal of Robotics Research*, vol. 35, no. 11, pp. 1354–1392, September 2016.